

# Diophantine properties of the *zeros* of (monic) polynomials the *coefficients* of which are the *zeros* of Hermite polynomials

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## Abstract

Let  $c_m$ , with  $m = 1, \dots, N$  (with  $N$  an arbitrary positive integer,  $N \geq 2$ ) be the  $N$  zeros (arbitrarily ordered) of the Hermite polynomial  $H_N(c)$ , of order  $N$  and argument  $c$ :  $H_N(c_m) = 0$ . Let the monic polynomial  $p_N(z)$  of degree  $N$  in the variable  $z$  be defined as follows:

$$p_N(z) = z^N + \sum_{m=1}^N (c_m z^{N-m}) = \prod_{n=1}^N (z - z_n) .$$

The first equality identifies the  $N$  *coefficients*  $c_m$  of this polynomial  $p_N(z)$  as the  $N$  *zeros*  $c_m$  of the Hermite polynomial of order  $N$ ; note that there are  $N!$  such *different* polynomials  $p_N(z)$ , depending on the ordering assignment of the  $N$  zeros  $c_m$  of the Hermite polynomial of order  $N$ . The second equality identifies (uniquely up to permutations) the  $N$  zeros  $z_n$  of the polynomial  $p_N(z)$ . In this paper we define in terms of these  $N$  zeros  $z_n$  two  $N \times N$  matrices  $\mathbf{M}^{(1)}$  and  $\mathbf{M}^{(2)}$  with *integer* respectively *square-integer* eigenvalues  $\lambda_m^{(1)} = m$  respectively  $\lambda_m^{(2)} = m^2$ ,  $m = 1, \dots, N$ . The technique whereby these findings are demonstrated can be extended to other named polynomials.

**Keywords:** zeros of polynomials; Hermite polynomials; special functions, Diophantine matrices.

**MSC:** 11C08, 70F10, 70K42, 11D41, 33E99.

## 1 Introduction

**Notation 1.1.** Unless otherwise indicated, hereafter  $N$  is an *arbitrary positive integer*,  $N \geq 2$ , indices such as  $n, m, \ell, j, k, \dots$  run over the *integers* from 1 to  $N$ , **boldface** lower-case letters indicate  $N$ -vectors (for instance the vector  $\mathbf{z}$  has the  $N$  components  $z_n$ ) and **boldface** upper-case letters indicate  $N \times N$  matrices (for instance the matrix  $\mathbf{M}$  has the  $N^2$  components  $M_{nm}$ ). The *imaginary unit* is hereafter denoted as  $\mathbf{i}$  ( $\mathbf{i}^2 = -1$ ; of course  $\mathbf{i}$  is not a  $N$ -vector!). For

quantities depending on the independent variable  $t$  a superimposed dot indicates differentiation with respect to  $t$ . The Kronecker symbol  $\delta_{nm}$  has the usual meaning:  $\delta_{nm} = 1$  if  $n = m$ ,  $\delta_{nm} = 0$  if  $n \neq m$ . And we adopt throughout the usual convention according to which a void sum vanishes and a void product equals unity:  $\sum_{j=J}^K f_j = 0$ ,  $\prod_{j=J}^K f_j = 1$  if  $J > K$ . Finally we introduce the following two convenient notations:

$$\sigma_j(\mathbf{z}) = \sum_{1 \leq s_1 < s_2 < \dots < s_j \leq N} z_{s_1} z_{s_2} \cdots z_{s_j} , \quad (1a)$$

$$\sigma_{m,j}(\mathbf{z}) = \sum_{1 \leq s_1 < s_2 < \dots < s_{j-1} \leq N ; s_k \neq m, k=1, \dots, j-1} z_{s_1} z_{s_2} \cdots z_{s_{j-1}} , \quad (1b)$$

where of course the symbol

$$\sum_{1 \leq s_1 < s_2 < \dots < s_j \leq N}$$

denotes the sum from 1 to  $N$  over the  $j$  integer indices  $s_1, s_2, \dots, s_j$  with the restriction that  $s_1 < s_2 < \dots < s_j$ , while the symbol

$$\sum_{1 \leq s_1 < s_2 < \dots < s_{j-1} \leq N ; s_k \neq m, k=1, \dots, j-1}$$

denotes the sum from 1 to  $N$  over the  $(j-1)$  indices  $s_1, s_2, \dots, s_{j-1}$  with the restriction that  $s_1 < s_2 < \dots < s_{j-1}$  and moreover the requirement that *all these indices be different from  $m$* . We note that  $\sigma_{m,1}(\mathbf{z}) = 0$  according to the convention (see above) that a sum over an empty set of indices equals zero. ■

A class of properties satisfied by the  $N$  zeros  $z_n$  of several named polynomials of order  $N$  has been obtained via the identification of  $N \times N$  matrices, constructed with these  $N$  zeros  $z_n$ , which feature eigenvalues displaying remarkable *Diophantine* properties: see for instance [1, 2]. A general technique to arrive at such results goes through the following two steps.

*First step.* A dynamical system is manufactured, characterized, say, by  $N$  Newtonian equations of motion (written as follows in  $N$ -vector notation),

$$\dot{\boldsymbol{\zeta}} = \mathbf{i} \mathbf{f}(\boldsymbol{\zeta}) , \quad \boldsymbol{\zeta} = \boldsymbol{\zeta}(t) , \quad (2a)$$

featuring the following two peculiar properties. (i) This dynamical system is *isochronous* with period  $2\pi$ : *all* its solutions are *completely periodic* with period  $2\pi$ ,

$$\boldsymbol{\zeta}(t + 2\pi) = \boldsymbol{\zeta}(t) . \quad (2b)$$

(ii) This dynamical system features the equilibrium configuration

$$\boldsymbol{\zeta}(t) = \mathbf{z} , \quad (3)$$

where the  $N$  components  $z_n$  of the  $N$ -vector  $\mathbf{z}$  are just the  $N$  zeros of the named polynomial under consideration.

*Second step.* The behavior of dynamical system (2a) in the *infinitesimal* neighborhood of its equilibrium (3) is investigated in the standard manner, by setting

$$\boldsymbol{\zeta}(t) = \mathbf{z} + \varepsilon \mathbf{v}(t) \quad (4)$$

with  $\varepsilon$  infinitesimal, and by thereby obtaining from (2a) the *linearized* system

$$\dot{\mathbf{v}}(t) = \mathbf{i} \mathbf{M}(\mathbf{z}) \mathbf{v}(t) , \quad (5a)$$

where of course the ( $t$ -independent)  $N \times N$  matrix  $\mathbf{M}(\mathbf{z})$  is defined componentwise as follows:

$$M_{nm}(\mathbf{z}) = \left. \frac{\partial f_n(\boldsymbol{\zeta})}{\partial \zeta_m} \right|_{\boldsymbol{\zeta}=\mathbf{z}} . \quad (5b)$$

Hence the solution of this linearized system (5) is a linear combination, with constant coefficients, of exponential functions  $\exp(\mathbf{i}\lambda_m t)$ , where  $\lambda_m$  are the  $N$  eigenvalues of the  $N \times N$  matrix  $\mathbf{M}(\mathbf{z})$ . But if the  $t$ -evolution of the  $N$  solutions  $\zeta_n(t)$  of the dynamical system (2a) are characterized by the periodicity property (2b), the  $N$  quantities  $v_n(t)$ —see (4)—must also possess this *same* property. Hence the  $N$  eigenvalues  $\lambda_m$  must feature the *Diophantine* property to be *integers*; and these *integers* can be actually identified by comparing the behavior of the *solvable* system with that of its *linearized* version in the *infinitesimal* and *immediate* vicinity of the relevant equilibria.

**Remark 1.1.** An analogous process can be applied to a *second-order* dynamical system  $\ddot{\boldsymbol{\zeta}} = \mathbf{f}(\boldsymbol{\zeta})$  satisfying conditions (i) and (ii), instead of the *first-order* system (2a). In this case, the linearization obtained in the *second step* leads to a *second-order* linear system  $\ddot{\mathbf{v}}(t) = -\mathbf{M}(\mathbf{z})\mathbf{v}(t)$ , where the matrix  $\mathbf{M}(\mathbf{z})$  is defined componentwise by  $M_{nm}(\mathbf{z}) = -\left. \frac{\partial f_n(\boldsymbol{\zeta})}{\partial \zeta_m} \right|_{\boldsymbol{\zeta}=\mathbf{z}}$ . ■

In this paper we again follow this approach, but with a new twist—based on a new development allowing to manufacture larger classes of *solvable* dynamical systems [3, 4]—yielding results which seem new and indeed somewhat surprising. Indeed, the  $N \times N$  matrix  $\mathbf{M}^{(1)}(\mathbf{z})$ , which we identify as having  $N$  *integer* eigenvalues, is defined in terms of the  $N$  *zeros*  $z_n$  of (monic) polynomials of degree  $N$  in  $z$  the  $N$  *coefficients* of which are the  $N$  *zeros* of Hermite polynomials of degree  $N$ . Moreover, by starting from the system of Newtonian (second-order) ODEs the solvability of which has been demonstrated in [4] rather than from a first-order system of ODEs such as (2a), see **Remark 1.1**, we provide an analogous derivation of another  $N \times N$  matrix  $\mathbf{M}^{(2)}(\mathbf{z})$ —again constructed with the same zeros  $z_n$ —featuring  $N$  *squared-integers* as eigenvalues. These findings are reported in the following Section 2 and proven in Section 3. Section 4 (“Outlook”) outlines tersely some possible further developments.

## 2 Results

In this section we report our findings; their proof is provided in the following Section 3.

**Proposition 2.1.** Let  $c_m$  with  $m = 1, \dots, N$  be the  $N$  zeros (arbitrarily ordered) of the Hermite polynomial  $H_N(c)$  of order  $N$  and argument  $c$  (see for instance [5]):

$$H_N(c) = N! \sum_{k=0}^{[[N/2]]} \left[ \frac{(-1)^k (2c)^{N-2k}}{k! (N-2k)!} \right], \quad (6a)$$

$$H_N(c_m) = 0, \quad m = 1, \dots, N, \quad (6b)$$

where of course the notation  $[[N/2]]$  denotes the integer part of  $N/2$ , i. e.  $N/2$  if  $N$  is *even*,  $(N-1)/2$  if  $N$  is *odd*. Let the monic polynomial  $p_N(z)$  of degree  $N$  in the variable  $z$  be defined as follows:

$$p_N(z) = z^N + \sum_{m=1}^N (c_m z^{N-m}) = \prod_{n=1}^N (z - z_n). \quad (7)$$

The first equality identifies the  $N$  *coefficients*  $c_m$  of this polynomial  $p_N(z)$  as the  $N$  *zeros*  $c_m$  of the Hermite polynomial of order  $N$ ; note that there are  $N!$  such *different* polynomials  $p_N(z) \equiv p_N^{(\mu)}(z)$ ,  $\mu = 1, \dots, N!$ , depending on the ordering assignment of the  $N$  zeros  $c_m$  of the Hermite polynomial of order  $N$ . The second equality identifies (*uniquely up to permutations*) the  $N$  zeros  $z_n^{(\mu)}$  of the polynomial  $p_N^{(\mu)}(z)$ .

Let the  $N \times N$  matrices  $\mathbf{M}^{(1)(\mu)} \equiv \mathbf{M}^{(1)}(\mathbf{z}^{(\mu)})$  be defined componentwise as follows in terms of the  $N$  zeros  $(z_1^{(\mu)}, \dots, z_N^{(\mu)}) \equiv \mathbf{z}^{(\mu)}$  of the polynomial  $p_N^{(\mu)}(z)$ :

$$M_{nm}^{(1)(\mu)} \equiv M_{nm}^{(1)}(\mathbf{z}^{(\mu)}) = - \left\{ \prod_{\ell=1, \ell \neq n}^N \frac{1}{[z_n^{(\mu)} - z_\ell^{(\mu)}]} \right\} \cdot \sum_{j=1}^N \left\{ [z_n^{(\mu)}]^{N-j} \cdot \left[ w_{j,m}^{(\mu)} + \sum_{s=1, s \neq j}^N \frac{w_{j,m}^{(\mu)} - w_{s,m}^{(\mu)}}{[c_j^{(\mu)} - c_s^{(\mu)}]^2} \right] \right\}, \quad (8a)$$

where

$$c_m^{(\mu)} \equiv c_m(\mathbf{z}^{(\mu)}) = (-1)^m \sigma_m(\mathbf{z}^{(\mu)}) \quad (8b)$$

and

$$w_{j,m}^{(\mu)} \equiv w_{j,m}(\mathbf{z}^{(\mu)}) = (-1)^j [\delta_{j1} + \sigma_{m,j}(\mathbf{z}^{(\mu)})] \quad (8c)$$

(see **Notation 1.1**).

Then the  $N$  eigenvalues  $\lambda_m^{(1)}$  of the  $N \times N$  matrices  $\mathbf{M}^{(1)(\mu)}$  are given by the following neat (*Diophantine*) formula:

$$\lambda_m^{(1)} = m, \quad m = 1, 2, \dots, N. \quad \blacksquare \quad (9)$$

**Remark 2.1.** Note that in **Proposition 2.1** all the *different*  $N!$  matrices  $\mathbf{M}^{(1)(\mu)} \equiv \mathbf{M}^{(1)}(\mathbf{z}^{(\mu)})$ , where  $\mu = 1, 2, \dots, N!$ , feature the *same* set of  $N$  eigenvalues  $\lambda_m^{(1)}$ . Also note that, although definition (8a) of the matrix  $\mathbf{M}^{(1)(\mu)}$  depends on the ordering assignment of the  $N$  zeros  $z_n^{(\mu)}$  of the polynomial  $p^{(\mu)}(z)$ ,

such different assignments produce the *same* matrix  $\mathbf{M}^{(1)(\mu)}$  up to a *reshuffling of its lines and columns* and have therefore no relevance for the eigenvalues of  $\mathbf{M}^{(1)(\mu)}$ : indeed, a switch of the two zeros  $z_n^{(\mu)}$  and  $z_m^{(\mu)}$  of the polynomial  $p^{(\mu)}(z)$  yields a new matrix  $\hat{\mathbf{M}}^{(1)(\mu)}$ , which corresponds to the original matrix  $\mathbf{M}^{(1)(\mu)}$  up to the exchange of the  $n$ -th and the  $m$ -th rows and columns of  $\mathbf{M}^{(1)(\mu)}$  (see also below **Remark 3.2**). ■

**Proposition 2.2.** Assume the notation of **Proposition 2.1**. If the  $N \times N$  matrices  $\mathbf{M}^{(2)(\mu)} = \mathbf{M}^{(2)}(\mathbf{z}^{(\mu)})$  are defined by

$$M_{nm}^{(2)(\mu)} \equiv M_{nm}^{(2)}(\mathbf{z}^{(\mu)}) = - \left\{ \prod_{\ell=1, \ell \neq n}^N \frac{1}{[z_n^{(\mu)} - z_\ell^{(\mu)}]} \right\} \cdot \sum_{j=1}^N \left\{ [z_n^{(\mu)}]^{N-j} \cdot \left[ w_{j,m}^{(\mu)} + 6 \sum_{s=1, s \neq j}^N \frac{w_{j,m}^{(\mu)} - w_{s,m}^{(\mu)}}{[c_j^{(\mu)} - c_s^{(\mu)}]^4} \right] \right\}, \quad (10)$$

where  $\mu = 1, 2, \dots, N!$ , then their eigenvalues are given by

$$\lambda_m^{(2)} = m^2, \quad m = 1, 2, \dots, N. \quad (11)$$

**Remark 2.2.** Note that in **Proposition 2.2** all the *different*  $N!$  matrices  $\mathbf{M}^{(2)(\mu)} \equiv \mathbf{M}^{(2)}(\mathbf{z}^{(\mu)})$ , where  $\mu = 1, 2, \dots, N!$ , feature the *same* set of  $N$  eigenvalues  $\lambda_m^{(2)}$ . Also note that, although definition (10) of the matrix  $\mathbf{M}^{(2)(\mu)}$  depends on the ordering assignment of the  $N$  zeros  $z_n^{(\mu)}$  of the polynomial  $p^{(\mu)}(z)$ , such different assignments produce the *same* matrix  $\mathbf{M}^{(2)(\mu)}$  up to a *reshuffling of its lines and columns* and have therefore no relevance for the eigenvalues of  $\mathbf{M}^{(2)(\mu)}$  (cf. **Remarks 2.1** and **3.2**). ■

**Examples 2.1** and **2.2** illustrate **Proposition 2.1** for the cases where  $N = 2$  and  $N = 3$ .

**Example 2.1.** Let us construct the  $2! = 2$  matrices  $\mathbf{M}^{(1)(\mu)}$  (see **Proposition 2.1**) for the case  $N = 2$ , where the index  $\mu \in \{1, 2\}$  identifies the 2 ordering assignments of the 2 zeros  $(c_1^{(\mu)}, c_2^{(\mu)}) = \left(-\frac{(-1)^\mu}{\sqrt{2}}, \frac{(-1)^\mu}{\sqrt{2}}\right)$  of the Hermite polynomial of degree 2,  $H_2(c) = 4c^2 - 2$ . In this special case the 2 numbers  $c_1^{(\mu)}, c_2^{(\mu)}$  enter in definition (8a) of the matrix  $\mathbf{M}^{(1)(\mu)}$  only via the expression  $[c_1^{(\mu)} - c_2^{(\mu)}]^{-2} = 1/2$ , which does not depend on the index  $\mu$  distinguishing the two different orderings of the two zeros  $c_1^{(\mu)}, c_2^{(\mu)}$  of the Hermite polynomial  $H_2(c)$ . Therefore, both choices of the pair  $(c_1^{(\mu)}, c_2^{(\mu)})$  yield the same formula for the matrix  $\mathbf{M}^{(1)(\mu)}$  in terms of the zeros  $\mathbf{z}^{(\mu)} = (z_1^{(\mu)}, z_2^{(\mu)})$  of the polynomial  $z^2 + c_1^{(\mu)}z + c_2^{(\mu)}$ :

$$\mathbf{M}^{(1)(\mu)} = \mathbf{M}^{(1)}(z_1^{(\mu)}, z_2^{(\mu)}) = \begin{pmatrix} \frac{3}{2} - \frac{1 - z_1^{(\mu)} z_2^{(\mu)}}{2[z_1^{(\mu)} - z_2^{(\mu)}]} & -\frac{1 - [z_1^{(\mu)}]^2}{2[z_1^{(\mu)} - z_2^{(\mu)}]} \\ \frac{1 - [z_2^{(\mu)}]^2}{2[z_1^{(\mu)} - z_2^{(\mu)}]} & \frac{3}{2} + \frac{1 - z_1^{(\mu)} z_2^{(\mu)}}{2[z_1^{(\mu)} - z_2^{(\mu)}]} \end{pmatrix}. \quad (12a)$$

Here  $z_1^{(\mu)}$  and  $z_2^{(\mu)}$  are of course the 2 zeros of the polynomial

$$p^{(\mu)}(z) = z^2 + \frac{(-1)^\mu (1-z)}{\sqrt{2}}, \quad \mu = 1, 2, \quad (12b)$$

hence

$$z_\pm^{(\mu)} = \frac{(-1)^\mu \pm \sqrt{1 - (-1)^\mu 4\sqrt{2}}}{2\sqrt{2}}, \quad \mu = 1, 2 \quad (12c)$$

(note that for  $\mu = 1$  these two zeros are *real*, for  $\mu = 2$  they are complex conjugate).

It can be easily verified that each of the matrices  $\mathbf{M}^{(1)(\mu)}$ ,  $\mu = 1, 2$ , given by formula (12a) has the two eigenvalues 1 and 2 consistently with **Proposition 2.1**. And it is easy to see from formula (12a) that the matrix  $\mathbf{M}^{(1)}(z_2^{(\mu)}, z_1^{(\mu)})$  can be obtained from the matrix  $\mathbf{M}^{(1)}(z_1^{(\mu)}, z_2^{(\mu)})$  by switching its rows and columns, consistently with **Remark 2.1**. ■

**Example 2.2.** For  $N = 3$  the 3 zeros of the Hermite polynomial  $H_3(c) = 8c^3 - 12c = 4c(2c^2 - 3)$  are  $0, \pm\sqrt{3/2}$ . So we make the following  $6 = 3!$  different assignments:

$$c_1^{(1)} = 0, \quad c_2^{(1)} = \sqrt{3/2}, \quad c_3^{(1)} = -\sqrt{3/2}, \quad (13a)$$

$$c_1^{(2)} = 0, \quad c_2^{(2)} = -\sqrt{3/2}, \quad c_3^{(2)} = \sqrt{3/2}, \quad (13b)$$

$$c_1^{(3)} = \sqrt{3/2}, \quad c_2^{(3)} = 0, \quad c_3^{(3)} = -\sqrt{3/2}, \quad (13c)$$

$$c_1^{(4)} = \sqrt{3/2}, \quad c_2^{(4)} = -\sqrt{3/2}, \quad c_3^{(4)} = 0, \quad (13d)$$

$$c_1^{(5)} = -\sqrt{3/2}, \quad c_2^{(5)} = \sqrt{3/2}, \quad c_3^{(5)} = 0, \quad (13e)$$

$$c_1^{(6)} = -\sqrt{3/2}, \quad c_2^{(6)} = 0, \quad c_3^{(6)} = \sqrt{3/2}; \quad (13f)$$

and correspondingly we define the following 6 polynomials  $p_3^{(\mu)}(z)$ , of third degree in  $z$ , and their sets of 3 zeros  $z_1^{(\mu)}, z_2^{(\mu)}, z_3^{(\mu)}$ :

$$p_3^{(\mu)}(z) = z^3 + \sum_{m=1}^3 [c_m^{(\mu)} z^{3-m}] = \prod_{n=1}^3 [z - z_n^{(\mu)}], \quad \mu = 1, \dots, 6. \quad (14)$$

The 3 zeros  $z_1^{(\mu)}, z_2^{(\mu)}, z_3^{(\mu)}$  can be easily obtained from this formula (if need be, via Mathematica), but rather than writing here their exact values (involving square and cubic roots) we simply provide their (of course approximate) numerical values in decimal form:

$$z_1^{(1)} = 0.7090, \quad z_2^{(1)} = -0.3545 - 1.2656 \mathbf{i}, \quad z_3^{(1)} = -0.3545 + 1.2656 \mathbf{i}, \quad (15a)$$

$$z_1^{(2)} = 0.7202 - 0.5758 \mathbf{i}, \quad z_2^{(2)} = 0.7202 + 0.5758 \mathbf{i}, \quad z_3^{(2)} = -1.4405, \quad (15b)$$

$$z_1^{(3)} = -1.0031 + 0.7492 \mathbf{i}, \quad z_2^{(3)} = -1.0031 - 0.7492 \mathbf{i}, \quad z_3^{(3)} = 0.7814, \quad (15c)$$

$$z_1^{(4)} = 0, \quad z_2^{(4)} = -1.8772, \quad z_3^{(4)} = 0.6524, \quad (15d)$$

$$z_1^{(5)} = 0, \quad z_2^{(5)} = -1.8772, \quad z_3^{(5)} = 0.6524, \quad (15e)$$

$$z_1^{(6)} = -0.7814, \quad z_2^{(6)} = 1.0031 - 0.7492 \mathbf{i}, \quad z_3^{(6)} = 1.0031 + 0.7492 \mathbf{i}. \quad (15f)$$

Of course these numbers are defined up to permutations, but, as explained below, this has no relevance for the *eigenvalues* of the matrices  $\mathbf{M}^{(1)}$  and  $\mathbf{M}^{(2)}$ , see **Remarks 2.1** and **3.2**. Note moreover that, as it happens, the zeros of the polynomials  $p_3^{(4)}(z)$  and  $p_3^{(5)}(z)$  are the same.

From formulas (8c) for  $N = 3$  we moreover obtain that

$$w_{1,m}^{(\mu)} = -1, \quad w_{2,m}^{(\mu)} = z_{m+1}^{(\mu)} + z_{m+2}^{(\mu)}, \quad w_{3,m}^{(\mu)} = -z_{m+1}^{(\mu)} z_{m+2}^{(\mu)}, \quad m = 1, 2, 3 \pmod{3}. \quad (16)$$

By inserting these values of the quantities  $w_{j,m}^{(\mu)}$  into formula (8a) for  $N = 3$  we obtain the components of the corresponding 6 matrices  $\mathbf{M}^{(1)(\mu)}$ :

$$\begin{aligned} M_{nm}^{(1)(\mu)} = & -\frac{1}{\left[z_n^{(\mu)} - z_{n+1}^{(\mu)}\right] \left[z_n^{(\mu)} - z_{n+2}^{(\mu)}\right]} \\ & \cdot \left\{ -\left[z_n^{(\mu)}\right]^2 + z_n^{(\mu)} \left[z_{m+1}^{(\mu)} + z_{m+2}^{(\mu)}\right] - z_{m+1}^{(\mu)} z_{m+2}^{(\mu)} \right. \\ & + \left[z_n^{(\mu)} - 1\right] \left( -\frac{z_n^{(\mu)} \left[1 + z_{m+1}^{(\mu)} + z_{m+2}^{(\mu)}\right]}{\left[c_1^{(\mu)} - c_2^{(\mu)}\right]^2} \right. \\ & + \frac{\left[-1 + z_{m+1}^{(\mu)} z_{m+2}^{(\mu)}\right] \left[z_n^{(\mu)} + 1\right]}{\left[c_1^{(\mu)} - c_3^{(\mu)}\right]^2} \\ & \left. \left. + \frac{\left[z_{m+1}^{(\mu)} + z_{m+2}^{(\mu)} + z_{m+1}^{(\mu)} z_{m+2}^{(\mu)}\right]}{\left[c_2^{(\mu)} - c_3^{(\mu)}\right]^2} \right) \right\}, \quad m = 1, 2, 3 \pmod{3}. \quad (17) \end{aligned}$$

It can be checked by direct computation that the 6 matrices given by (17) *all* feature the 3 eigenvalues 1, 2, 3. And it is again plain that a permutation of the zeros of the polynomial  $p_3^{(\mu)}(z)$  results in a corresponding permutation of the rows and the columns of the matrix  $\mathbf{M}^{(1)(\mu)}$ , see **Remarks 2.1** and **3.2**. ■

Let us end this section by displaying the matrices  $\mathbf{M}^{(2)}(\mathbf{z})$ , defined in **Proposition 2.2**, for  $N = 2$  and  $N = 3$ . Note that these matrices  $\mathbf{M}^{(2)}(\mathbf{z})$  do *not* coincide with the matrices  $[\mathbf{M}^{(1)}(\mathbf{z})]^2$  (see those reported in **Examples 2.1** and **2.2**), although they of course feature the *same* squared-integer eigenvalues.

**Example 2.3.** Let us construct the  $2! = 2$  matrices  $\mathbf{M}^{(2)(\mu)}$  for the case  $N = 2$ , where, as in **Example 2.1**, the index  $\mu \in \{1, 2\}$  identifies the 2 ordering assignments of the 2 zeros  $\left(c_1^{(\mu)}, c_2^{(\mu)}\right) = \left(-\frac{(-1)^\mu}{\sqrt{2}}, \frac{(-1)^\mu}{\sqrt{2}}\right)$  of the Hermite polynomial of degree 2,  $H_2(c) = 4c^2 - 2$ . Similarly to **Example 2.1**, in this special

case the 2 numbers  $c_1^{(\mu)}, c_2^{(\mu)}$  enter in definition (10) of the matrix  $\mathbf{M}^{(2)(\mu)}$  only via the expression  $(c_1^{(\mu)} - c_2^{(\mu)})^{-4} = 1/4$ , which does not depend on the index  $\mu$  distinguishing the two different orderings of the two zeros  $c_1^{(\mu)}, c_2^{(\mu)}$  of the Hermite polynomial  $H_2(c)$ . Therefore both choices of the pair  $(c_1^{(\mu)}, c_2^{(\mu)})$  yield the same formula for the matrix  $\mathbf{M}^{(\mu)}$  in terms the zeros  $\mathbf{z}^{(\mu)} = (z_1^{(\mu)}, z_2^{(\mu)})$  of the polynomial  $z^2 + c_1^{(\mu)}z + c_2^{(\mu)}$ :

$$\mathbf{M}^{(2)(\mu)} = \mathbf{M}^{(2)}(z_1^{(\mu)}, z_2^{(\mu)}) = \begin{pmatrix} \frac{5}{2} - \frac{3}{2} \frac{1 - z_1^{(\mu)}}{z_1^{(\mu)} - z_2^{(\mu)}} \frac{z_2^{(\mu)}}{z_1^{(\mu)} - z_2^{(\mu)}} & -\frac{3}{2} \frac{1 - (z_1^{(\mu)})^2}{z_1^{(\mu)} - z_2^{(\mu)}} \\ \frac{3}{2} \frac{1 - (z_2^{(\mu)})^2}{z_1^{(\mu)} - z_2^{(\mu)}} & \frac{5}{2} + \frac{3}{2} \frac{1 - z_1^{(\mu)}}{z_1^{(\mu)} - z_2^{(\mu)}} \frac{z_2^{(\mu)}}{z_1^{(\mu)} - z_2^{(\mu)}} \end{pmatrix}. \quad (18)$$

Here  $z_1^{(\mu)}$  and  $z_2^{(\mu)}$  are of course the 2 zeros of the polynomial (12b) given by (12c) (note, again, that for  $\mu = 1$  these two zeros are *real*, for  $\mu = 2$  they are complex conjugate).

It can be easily verified that each of the matrices  $\mathbf{M}^{(2)(\mu)}$ ,  $\mu = 1, 2$ , given by formula (18) has the two eigenvalues 1 and 4, consistently with **Proposition 2.2**. And it is easy to see from formula (18) that the matrix  $\mathbf{M}^{(2)}(z_2^{(\mu)}, z_1^{(\mu)})$  can be obtained from the matrix  $\mathbf{M}^{(2)}(z_1^{(\mu)}, z_2^{(\mu)})$  by switching its rows and columns, consistently with **Remark 2.2**. ■

**Example 2.4.** For  $N = 3$  the 3 zeros of the Hermite polynomial  $H_3(c) = 8c^3 - 12c = 4c(2c^2 - 3)$  are  $0, \pm\sqrt{3}/2$ . As in **Example 2.2**, we make  $3! = 6$  different assignments of the coefficients  $c_1^{(\mu)}, c_2^{(\mu)}, c_3^{(\mu)}$ , given by (13), and define the polynomials  $p_3^{(\mu)}(z)$  by (14). The 3 zeros  $z_1^{(\mu)}, z_2^{(\mu)}, z_3^{(\mu)}$  can be easily obtained from (14); their approximate numerical values are given by (15). Of course these numbers are defined up to permutations, but, as explained below, this has no relevance for the *eigenvalues* of the matrices  $\mathbf{M}^{(1)}$  and  $\mathbf{M}^{(2)}$ , see **Remark 2.2**.

For  $N = 3$  the coefficients  $w_{j,m}^{(\mu)}$  from (8c) are given by (16). By inserting these values of the quantities  $w_{j,m}^{(\mu)}$  into formula (10) for  $N = 3$  we obtain the



components of the corresponding 6 matrices  $\mathbf{M}^{(2)(\mu)}$ :

$$\begin{aligned}
M_{nm}^{(2)(\mu)} = & -\frac{1}{\left[z_n^{(\mu)} - z_{n+1}^{(\mu)}\right] \left[z_n^{(\mu)} - z_{n+2}^{(\mu)}\right]} \\
& \cdot \left\{ -\left[z_n^{(\mu)}\right]^2 + z_n^{(\mu)} \left[z_{m+1}^{(\mu)} + z_{m+2}^{(\mu)}\right] - z_{m+1}^{(\mu)} z_{m+2}^{(\mu)} \right. \\
& + 6 \left[z_n^{(\mu)} - 1\right] \left( -\frac{z_n^{(\mu)} \left[1 + z_{m+1}^{(\mu)} + z_{m+2}^{(\mu)}\right]}{\left[c_1^{(\mu)} - c_2^{(\mu)}\right]^4} \right. \\
& + \frac{\left[-1 + z_{m+1}^{(\mu)} z_{m+2}^{(\mu)}\right] \left[z_n^{(\mu)} + 1\right]}{\left[c_1^{(\mu)} - c_3^{(\mu)}\right]^4} \\
& \left. \left. + \frac{\left[z_{m+1}^{(\mu)} + z_{m+2}^{(\mu)} + z_{m+1}^{(\mu)} z_{m+2}^{(\mu)}\right]}{\left[c_2^{(\mu)} - c_3^{(\mu)}\right]^4} \right) \right\}, \quad m = 1, 2, 3 \pmod{3}. \quad (19)
\end{aligned}$$

It can be checked by direct computation that the 6 matrices given by (19) *all* feature the 3 eigenvalues 1, 4, 9. Moreover, it can be verified, again by computation, that a permutation of the zeros of the polynomial  $p_3^{(\mu)}(z)$  results in an appropriate permutation of the rows and the columns of the matrix  $\mathbf{M}^{(2)(\mu)}$ , see **Remark 2.2** . ■

### 3 Proofs

In this Section 3 we prove **Propositions 2.1** and **2.2**.

**Proof of Proposition 2.1.** Our starting point is the system of  $N$  ODEs

$$\dot{\zeta}_n = - \left\{ \left[ \prod_{\ell=1, \ell \neq n}^N (\zeta_n - \zeta_\ell) \right]^{-1} \sum_{m=1}^N \left[ \dot{\gamma}_m (\zeta_n)^{N-m} \right] \right\} \quad (20a)$$

(see eq. (9a) of [3]), with

$$\dot{\gamma}_m = \mathbf{i} \left[ \gamma_m - \sum_{\ell=1, \ell \neq m}^N (\gamma_m - \gamma_\ell)^{-1} \right]. \quad (20b)$$

Above and hereafter we assume the  $2N$  quantities  $\zeta_n \equiv \zeta_n(t)$  and  $\gamma_m \equiv \gamma_m(t)$  to be functions of the independent variable  $t$ , and we denote with superimposed dots differentiations with respect to this variable (the dependence on which is, for notational simplicity, not always *explicitly* displayed, see for instance (20)). We moreover assume [3, 4] the quantities  $\gamma_m \equiv \gamma_m(t)$  respectively  $\zeta_n \equiv \zeta_n(t)$

to be the  $N$  *coefficients* respectively the  $N$  *zeros* of a time-dependent monic polynomial  $\psi_N(\zeta; t)$  of degree  $N$  in the variable  $\zeta$ :

$$\psi_N(\zeta; t) = \zeta^N + \sum_{m=1}^N \left[ \gamma_m(t) \zeta^{N-m} \right] = \prod_{n=1}^N [\zeta - \zeta_n(t)] , \quad (21a)$$

implying

$$\gamma_m(t) = (-1)^m \left[ \sum_{1 \leq s_1 < s_2 < \dots < s_m \leq N} \zeta_{s_1}(t) \zeta_{s_2}(t) \cdots \zeta_{s_m}(t) \right] . \quad (21b)$$

It is this connection that justifies the validity of (20a): see [3, 4].

It is on the other hand well known (see for instance [6] sect. 2.3.4.1) that dynamical system (20b) is *integrable* indeed *solvable* and *isochronous* with period  $2\pi$ , *all* its solutions featuring the remarkable property

$$\gamma_m(t + 2\pi) = \gamma_m(t) . \quad (22)$$

Hence, for the dynamical system implied by (20a) with (20b), reading

$$\dot{\zeta}_n = -\mathbf{i} \left[ \prod_{\ell=1, \ell \neq n}^N (\zeta_n - \zeta_\ell) \right]^{-1} \sum_{m=1}^N \left\{ \left[ \gamma_m - \sum_{\ell=1, \ell \neq m}^N (\gamma_m - \gamma_\ell)^{-1} \right] (\zeta_n)^{N-m} \right\} , \quad (23a)$$

with  $\gamma_m \equiv \gamma_m(t)$  expressed in terms of the  $N$  zeros  $\zeta_n \equiv \zeta_n(t)$  via (21b), all the solutions  $\zeta_n(t)$  are likewise periodic with period  $2\pi$ , being the  $N$  *zeros* of a polynomial  $\psi_N(\zeta; t)$ , of degree  $N$  in  $\zeta$  (see (21a)), which is itself periodic with period  $2\pi$  since its coefficients are *all* periodic with period  $2\pi$ :

$$\zeta_n(t + 2\pi) = \zeta_n(t) . \quad (23b)$$

**Remark 3.1.** It might be observed that the zeros  $\zeta_n(t)$  of a time-dependent polynomial of degree  $N$  in  $\zeta$  which is itself periodic with period  $T$  might themselves be periodic with a period which is a (generally small; see [8]) *integer multiple* of  $T$ , due to the possibility that the zeros, as it were, *exchange their roles* through their time evolution. But this is certainly not the case for a time evolution in which each zero evolves periodically remaining infinitely close to its equilibrium position in an equilibrium configuration of system (23); which is the case we consider below. ■

It is moreover clear that an equilibrium configuration of this dynamical system, (23a), is provided by the values  $\zeta_n(t) = z_n$  which are the zeros of the polynomial (21a) corresponding to the equilibrium configuration  $\gamma_n(t) = c_n$  of the system (20b), where the  $N$  quantities  $c_n$  satisfy the system of algebraic equations

$$c_m - \sum_{\ell=1, \ell \neq m}^N (c_m - c_\ell)^{-1} = 0 , \quad m = 1, \dots, N . \quad (24a)$$

It is on the other hand well known (see for instance Appendix C of [6]; this finding is actually much older, see for instance [9] and references therein) that the  $N$  zeros  $c_n$  of the Hermite polynomial of order  $N$  in  $c$ ,

$$H_N(c_n) = 0, \quad n = 1, \dots, N, \quad (24b)$$

satisfy this system of algebraic equations, (24a).

Next, let us look at the behavior of dynamical system (23) in the infinitesimal vicinity of the equilibrium configuration  $\zeta_n = z_n$ , where the  $N$  coordinates  $z_n$  are the  $N$  zeros of the polynomial  $p_N(z)$ , see (7). To this end we set

$$\zeta_n(t) = z_n + \varepsilon v_n(t), \quad \gamma_m(t) = c_m + \varepsilon w_m(t) + O(\varepsilon^2), \quad (25a)$$

with  $\varepsilon$  infinitesimal; and we note that, via (24a) with (21b), these two formulas, (25a), imply that

$$\begin{aligned} & \gamma_m - \sum_{\ell=1, \ell \neq m}^N (\gamma_m - \gamma_\ell)^{-1} \\ &= \varepsilon \left[ w_m + \sum_{\ell=1, \ell \neq m}^N (c_m - c_\ell)^{-2} (w_m - w_\ell) \right] + O(\varepsilon^2), \end{aligned} \quad (25b)$$

where

$$w_j(t) = (-1)^j \left\{ \sum_{1 \leq s_1 < s_2 < \dots < s_j \leq N} \left[ \sum_{q=1}^j v_{s_q}(t) \prod_{r=1, r \neq q}^j (z_{s_r}) \right] \right\}. \quad (25c)$$

It is then easily seen – via these formulas and the insertion of (25a) in (23) – that the dependent variables  $v_n \equiv v_n(t)$  evolve in time according to the following linearized version of (23):

$$\dot{v}_n(t) = -\mathbf{i} \left[ \prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell) \right]^{-1} \quad (26)$$

$$\cdot \left\{ \sum_{j=1}^N (z_n)^{N-j} \left[ w_j(t) + \sum_{\ell=1, \ell \neq j}^N \frac{w_j(t) - w_\ell(t)}{(c_j - c_\ell)^2} \right] \right\}, \quad (27)$$

where of course the quantities  $w_n(t)$  must be replaced by their expressions (25c).

Clearly this system can be more compactly rewritten as follows:

$$\dot{\mathbf{v}}(t) = \mathbf{i} \mathbf{M}^{(1)} \mathbf{v}(t), \quad (28)$$

with the time-independent  $N \times N$  matrix  $\mathbf{M}^{(1)}$  defined componentwise by (8).

**Remark 3.2.** It is easy to see that a switch between the two zeros  $z_n$  and  $z_m$  of the Hermite polynomial  $H_N(c)$  in the derivation of system (27), see (25a),

will result in  $v_n(t)$  and  $v_m(t)$  exchanging their roles. This, in turn, will result in a switch of the  $n$ -th and the  $m$ -th rows as well as the  $n$ -th and the  $m$ -th columns of the matrix  $\mathbf{M}^{(1)}$  in system (28), see also (8) and **Remark 2.1**. ■

System (28) implies that the  $N$ -vector  $\mathbf{v}(t)$  evolves as follows:

$$\mathbf{v}(t) = \sum_{m=1}^N \left[ a_m \exp(\mathbf{i} \lambda_m t) \mathbf{u}^{(m)} \right] , \quad (29a)$$

where  $\mathbf{u}^{(m)}$  respectively  $\lambda_m$  are the eigenvectors respectively the eigenvalues of the  $N \times N$  matrix  $\mathbf{M}^{(1)}$  (see (8)), and the  $N$  parameters  $a_m$  are of course characterized by the initial values of the  $N$ -vectors  $\mathbf{v}$  so that

$$\mathbf{v}(0) = \sum_{m=1}^N \left[ a_m \mathbf{u}^{(m)} \right] . \quad (29b)$$

But we know that the time evolution of the  $N$  coordinates  $\zeta_n(t)$  is periodic with period  $2\pi$ , see (23b), hence (see the first of the two formulas (25a)) the  $N$ -vector  $\mathbf{v}(t)$ , of components  $v_n(t)$ , must also be periodic with period  $2\pi$ . Hence—see (29a)—the  $N$  eigenvalues  $\lambda_m$  of the  $N \times N$  matrix  $\mathbf{M}^{(1)}$  must have the integer values  $m$  identified in **Proposition 2.1**, see (9). Q. E. D.

**Proof of Proposition 2.2.** Here we derive an analogous result to that reported in **Proposition 1.1**—identifying thereby a  $N \times N$  matrix  $\mathbf{M}^{(2)}(\mathbf{z})$  featuring as its  $N$  eigenvalues the *squared integers*  $m^2$ . The derivation of this finding—which was actually the first one we obtained, as follow-up to [4]—is reported below rather tersely, since it is quite analogous to the proof of **Proposition 1.1**, see above. Also the notation is generally identical to that used above, although the quantities used below should not be identified with those used above; only in some case we have appended the upper symbol <sup>(2)</sup>, to emphasize the difference of the results reported below from those of **Proposition 2.1** and its proof.

Our starting point is the novel dynamical system characterized by the  $N$  Newtonian equations of motion of goldfish type (see [4])

$$\ddot{\zeta}_n = \sum_{\ell=1, \ell \neq n}^N \left( \frac{2 \dot{\zeta}_n \dot{\zeta}_\ell}{\zeta_n - \zeta_\ell} \right) - \left\{ \left[ \prod_{\ell=1, \ell \neq n}^N (\zeta_n - \zeta_\ell) \right]^{-1} \sum_{m=1}^N \left[ \ddot{\gamma}_m (\zeta_n)^{N-m} \right] \right\} , \quad (30a)$$

with

$$\ddot{\gamma}_m = -\gamma_m + 2 \sum_{\ell=1, \ell \neq m}^N (\gamma_m - \gamma_\ell)^{-3} . \quad (30b)$$

We again assume, as above, that the quantities  $\zeta_n \equiv \zeta_n(t)$  respectively  $\gamma_m \equiv \gamma_m(t)$  are the  $N$  *zeros* respectively the  $N$  *coefficients* of a time-dependent monic polynomial  $\psi_N(\zeta; t)$  of degree  $N$  in the variable  $\zeta$ , see (21).

It is on the other hand well known that dynamical system (30b) is *integrable* indeed *solvable* and *isochronous* with period  $2\pi$ , *all* its solutions featuring the remarkable property

$$\gamma_m(t + 2\pi) = \gamma_m(t) \quad (31)$$

(see for instance [4, 6, 7]). Hence, for the dynamical system implied by (30a) with (30b), reading

$$\ddot{\zeta}_n = \sum_{\ell=1, \ell \neq n}^N \left( \frac{2 \dot{\zeta}_n \dot{\zeta}_\ell}{\zeta_n - \zeta_\ell} \right) - \left[ \prod_{\ell=1, \ell \neq n}^N (\zeta_n - \zeta_\ell) \right]^{-1} \cdot \sum_{m=1}^N \left\{ \left[ -\gamma_m + 2 \sum_{\ell=1, \ell \neq m}^N (\gamma_m - \gamma_\ell)^{-3} \right] (\zeta_n)^{N-m} \right\}, \quad (32)$$

with  $\gamma_m \equiv \gamma_m(t)$  expressed in terms of the  $N$  zeros  $\zeta_n \equiv \zeta_n(t)$  via (21b), all the solutions  $\zeta_n(t)$  are likewise periodic with period  $2\pi$ , being the  $N$  zeros of a polynomial  $\psi_N(\zeta; t)$ , of degree  $N$  in  $\zeta$  (see (21a)), the coefficients of which are *all* periodic with period  $2\pi$ :

$$\zeta_n(t + 2\pi) = \zeta_n(t). \quad (33)$$

Indeed an equilibrium configuration of dynamical system (32) is provided by the  $N$  zeros  $z_n$  of a polynomial  $p_N(z)$  the  $N$  coefficients  $c_m$  of which satisfy the set of  $N$  algebraic equations

$$-c_m + 2 \sum_{\ell=1, \ell \neq m}^N (c_m - c_\ell)^{-3} = 0; \quad (34)$$

since clearly the right-hand sides of the  $N$  ODEs (32) all vanish at equilibrium—i. e. when all the “velocities”  $\dot{\zeta}_n$  vanish—provided moreover  $\gamma_m = c_m$ ,  $m = 1, \dots, N$  with the  $N$  parameters  $c_m$  satisfying the  $N$  algebraic equations (34). But it is well known (see for instance [1, 6]) that the  $N$  zeros  $c_n$  of the Hermite polynomial  $H_N(c)$  do satisfy the set of  $N$  algebraic equations (34) (in addition to satisfying the different set of  $N$  algebraic equations (24a)).

Next, let us look at the behavior of dynamical system (32) in the infinitesimal vicinity of the equilibrium configuration  $\zeta_n = z_n$ , where the  $N$  coordinates  $z_n$  are the  $N$  zeros of the polynomial  $p_N(z)$ , see (7). Then via (25a) we get

$$\begin{aligned} & -\gamma_m + 2 \sum_{\ell=1, \ell \neq m}^N (\gamma_m - \gamma_\ell)^{-3} \\ &= -\varepsilon \left[ w_m + 6 \sum_{\ell=1, \ell \neq m}^N (c_m - c_\ell)^{-4} (w_m - w_\ell) \right] + O(\varepsilon^2), \end{aligned} \quad (35)$$

of course with  $w_j(t)$  defined by (25c).

It is then easily seen—via these formulas—that the insertion of (25a) in (32) implies that the dependent variables  $v_n \equiv v_n(t)$  evolve in time according to the following linearized version of (32):

$$\ddot{v}_n^{(2)}(t) = - \left[ \prod_{\ell=1, \ell \neq n}^N (z_n - z_\ell) \right]^{-1} \cdot \sum_{j=1}^N (z_n)^{N-j} \left\{ w_j(t) + 6 \sum_{\ell=1, \ell \neq j}^N \frac{w_j(t) - w_\ell(t)}{(c_j - c_\ell)^4} \right\}, \quad (36)$$

where the quantities  $w_n(t)$  must be replaced by their expressions (25c) (of course with  $v_m(t)$  replaced by  $v_m^{(2)}(t)$ ).

Clearly this system can be more compactly rewritten as follows:

$$\ddot{\mathbf{v}}^{(2)}(t) = -\mathbf{M}^{(2)} \mathbf{v}^{(2)}(t), \quad (37)$$

with the time-independent  $N \times N$  matrix  $\mathbf{M}^{(2)}$  defined by (10).

System (37) implies that the  $N$ -vector  $\mathbf{v}(t)$  evolves as follows:

$$\mathbf{v}^{(2)}(t) = \sum_{m=1}^N \left\{ \left[ a_m^{(2)} \cos(\lambda_m^{(2)} t) + b_m^{(2)} \frac{\sin(\lambda_m^{(2)} t)}{\lambda_m^{(2)}} \right] \mathbf{u}^{(2)(m)} \right\}, \quad (38a)$$

where  $\mathbf{u}^{(2)(m)}$  respectively  $[\lambda_m^{(2)}]$  are the eigenvectors respectively the eigenvalues of the  $N \times N$  matrix  $\mathbf{M}^{(2)}$ , and the  $2N$  parameters  $a_m^{(2)}$  and  $b_m^{(2)}$  are of course characterized by the initial values of the two  $N$ -vectors  $\mathbf{v}$  and  $\dot{\mathbf{v}}$  so that

$$\mathbf{v}^{(2)}(0) = \sum_{m=1}^N \left[ a_m^{(2)} \mathbf{u}^{(2)(m)} \right], \quad \dot{\mathbf{v}}^{(2)}(0) = \sum_{m=1}^N \left[ b_m^{(2)} \mathbf{u}^{(2)(m)} \right]. \quad (38b)$$

But we know that the time evolution of the  $N$  coordinates  $\zeta_n(t)$  is periodic with period  $2\pi$ , see (33), hence (see the first of the two formulas (25a)) the  $N$ -vector  $\mathbf{v}(t)$ , of components  $v_n(t)$ , must also be periodic with period  $2\pi$ . Hence—see (38a)—the  $N$  eigenvalues  $\lambda_m^{(2)}$  of the  $N \times N$  matrix  $\mathbf{M}^{(2)}(\mathbf{z})$  must have the squared-integer values  $m^2$ ,  $m = 1, 2, \dots, N$ . Q.E.D.

## 4 Outlook

The findings reported in this paper suggest some further developments, which we intend to pursue in future publications. A quite natural one is the extension of these results to other, more general, classes of polynomials than Hermite polynomials (see, for instance, the properties of the zeros of polynomials reported in [1, 2]). Another avenue of extension that we plan to explore is by iterating the

approach used in this paper, to find properties of the *zeros* of monic polynomials the *coefficients* of which are the *zeros* of monic polynomials the *coefficients* of which are the *zeros* of, say, Hermite polynomials; with an obviously ample choice of how this procedure can be further iterated and also combined with findings involving other named polynomials than Hermite polynomials.

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